

Casimir force between dielectric media with free charges

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The statistical mechanical approach to Casimir problems for dielectrics separated by a vacuum gap turns out to be compact and effective. A central ingredient of this method is the effect of interacting fluctuating dipole moments of the polarizable particles. At arbitrary temperature the path-integral formulation of quantized particles, developed by Høye-Stell and others, is needed. At high temperature—the limit considered in the present paper—the classical theory is, however, sufficient. Our present theory is related to an idea put forward earlier by Jancovici and Šamaj (2004), namely, to evaluate the Casimir force between parallel plates invoking an electronic plasma model and the Debye-Hückel theory for electrolytes. Their result was recently recovered by Høye (2008), using a related statistical mechanical method. In the present paper we generalize this by including a constant permittivity in the description. The present paper generalizes our earlier theory for parallel plates (1998), as well as for spherical dielectrics (2001). We also consider the Casimir force between a polarizable particle and a conductor with a small density of charges, finding agreement with the result recently derived by Pitaevskii (2008).

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I. INTRODUCTION

The large interest in the Casimir effect in recent years has resulted in a flood of papers, most of which are concerned with the field theoretical approach to the problem. This is quite natural, and is in accordance with the spirit in the original of Casimir paper [1]. Some general reviews of the Casimir effect can be found in Refs. [2–5]. Quantum field theoretical methods have proven to be quite effective even in the presence of dielectric media although there are fundamental problems here, especially regarding perfect conductor boundary conditions at sharp surfaces [6].

Now there exists an alternative and probably less known alternative route to derive a theory for the Casimir effect, namely, to start from statistical mechanics and regard the effect as due to interacting fluctuating dipole moments of polarizable particles. As shown by Brevik and Høye, the Casimir force between a pair of polarizable point particles can be recovered [7]. In this context, the path-integral formulation of quantized particle systems was utilized. The method had earlier been applied to polarizable fluid systems, by Høye and Stell, and Thomson *et al.* [8]. Subsequent generalizations led to the well-known Lifshitz formula for parallel plates [9]. Similar evaluations were performed in Refs. [10–12].

We wish to emphasize that, as a general remark, the quantum statistical mechanical method is quite compact and effective. For instance, the Casimir free energy due to the dispersion force between two polarizable particles as given by Eq. (5.15) in [7] is derived in a way that according to our view is simpler than the conventional field theoretical

method. The latter involves fourth-order approximation of perturbation theory [13]. Moreover, we ought to point out that the statistical mechanical approach opens new perspectives in the sense that one avoids quantizing the electromagnetic field itself. Instead, the field is playing the role of an agent permitting the interaction between polarizable particles. The two pictures are physically equivalent [7,9,11,12].

The topic with which we will be concerned with below is related to earlier work of Jancovici and Šamaj [14]. They realized that it should be possible to evaluate the Casimir force between metallic plates by employing an electronic plasma model. They considered accordingly charged particles at low density in a neutralized background, limiting themselves to the classical (i.e., the high-temperature limit) for which the Debye-Hückel theory of electrolytes is fully applicable. Calculating the pair-correlation function, and from that the local ionic density at the surfaces of the plates, they were able to recover the conventional Casimir result for real metals in the high-temperature limit. The issue of temperature corrections to the Casimir force has been subject to a lively discussion in the contemporary literature [15]. The ionic plasma has also been extended to the quantum-mechanical case by use of the path-integral formalism from a statistical mechanical viewpoint [16].

The recent paper of Høye [17] reconsidered the ionic plasma in the classical limit, using the statistical mechanical method in a different way to obtain the Casimir force. The correlation function was used to directly evaluate the average force between pairs of particles in the two plates and then the total force was found by integration. This is the same method used in Refs. [7,9]. The result was found to be in agreement with Ref. [14]. A notable feature of this approach is that it demonstrates how the modification of the density profile at the surface is a perturbing effect that can be neglected to leading order.

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In Ref. [17], the two parallel plates (or slabs) were assumed to be filled with ions but no dielectric properties due to the medium itself were envisaged. The present paper presents an extension and generalization of the theory in [17], in the sense that the dielectric properties are included. That means, we introduce permittivities ε in the slabs. The two permittivities are taken to be constant and equal. Magnetic properties in the media are omitted. In the gap region between the plates, we assume a vacuum. Our method consists in replacing polarizations by equivalent charges. That is, the polarization \mathbf{P} is taken equivalent to an induced charge density

$$\rho_l = -\nabla \cdot \mathbf{P}. \quad (1)$$

This follows from the connection between the electric field \mathbf{E} , the polarization \mathbf{P} , and the charge density ρ_c ,

$$\nabla \cdot \mathbf{E} + 4\pi \nabla \cdot \mathbf{P} = 4\pi \rho_c. \quad (2)$$

When evaluating the Casimir force based upon the ionic fluid correlation function, the induced charge can be considered as a free charge. Accordingly, no separate charged dipole and dipole-dipole correlation functions are needed. In addition, the dielectric properties are expressed via the influence of the permittivity upon the charge-charge correlation function.

In the next section we establish the basic formalism for the pair-correlation function and thereafter, in Sec. III, derive the Casimir force as a generalization of the expression found earlier for the case where charges were absent [9]. In Sec. IV we consider the force between a polarizable particle and a slab (half space) containing free charges. The treatment in that section is related to recent work of Pitaevskii [18]. Finally, in Sec. V we consider the Casimir interaction between spherical dielectric shells and extend the theory given earlier in Ref. [11] to the case where there are free charges present.

II. CORRELATION FUNCTION

To obtain the correlation function we start from the Ornstein-Zernike equation [17].

$$h(\mathbf{r}_2, \mathbf{r}_1) = c(\mathbf{r}_2, \mathbf{r}_1) + \int c(\mathbf{r}_2, \mathbf{r}') \rho(\mathbf{r}') h(\mathbf{r}', \mathbf{r}_1) d\mathbf{r}', \quad (3)$$

here extended to nonhomogeneous fluids. In this equation $h(\mathbf{r}_2, \mathbf{r}_1)$ is the correlation function, $c(\mathbf{r}_2, \mathbf{r}_1)$ is the direct correlation function, and $\rho(\mathbf{r}')$ is the number density of charges.

It may be appropriate to give a brief account of the background for this equation. The correlation function or pair-correlation function $h(\mathbf{r}_2, \mathbf{r}_1)$ is related to the pair distribution function $g(\mathbf{r}_2, \mathbf{r}_1)$ through

$$\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)h(\mathbf{r}_2, \mathbf{r}_1) = g(\mathbf{r}_2, \mathbf{r}_1) - \rho(\mathbf{r}_1)\rho(\mathbf{r}_2).$$

The $g(\mathbf{r}_2, \mathbf{r}_1)$ is the probability density for one particle to occupy the position \mathbf{r}_2 while another particle occupies the position \mathbf{r}_1 . For an ideal gas with uncorrelated particles, $g = \rho(\mathbf{r}_1)\rho(\mathbf{r}_2)$. Thus the h expresses deviations from the ideal-gas value. For a uniform fluid $\rho(\mathbf{r}) = \rho = \text{constant}$, and $h \rightarrow h(\mathbf{r}_2 - \mathbf{r}_1)$. The above equation as originally introduced by Ornstein and Zernike [19] serves as a definition of the direct

correlation function $c(\mathbf{r}_2, \mathbf{r}_1)$. In their investigation they noted that the direct correlation function c for a fluid was closely related to the pair interaction itself. In view of this, Eq. (3) can be given a simple interpretation: the resulting pair-correlation function h is the result of a direct correlation c plus correlations via other particles as expressed by the integral.

For weak long-range forces, the $c(\mathbf{r}, \mathbf{r}')$ is to leading order related to the interaction ψ in a simple way [20],

$$c(\mathbf{r}, \mathbf{r}') = -\beta \psi(\mathbf{r}, \mathbf{r}'), \quad (4)$$

where $\beta = 1/k_B T$, T is the temperature, and k_B is Boltzmann's constant. This follows from the γ ordering studied in Ref. [20], where γ is the inverse range of interaction and the limit $\gamma \rightarrow 0$ is considered. In the present case this becomes exact for large separations $\mathbf{r} - \mathbf{r}'$ with particles located in separate plates. However, for particles at close separation (in the same plate) there will be deviations. For low densities they can be neglected, in accordance with the Debye-Hückel theory for electrolytes. For higher densities such deviations will mainly change the inverse Debye-shielding length. This contributes only to a minor change in the effective separation between the plates [17]. We find no reason to consider this further here, especially since we consider semiconductors with low density of charges.

As the interaction follows from the electrostatic potential between two charges, we have

$$\nabla^2 c(\mathbf{r}, \mathbf{r}') = 4\pi \beta q_c^2 \frac{1}{\varepsilon(\mathbf{r}')} \delta(\mathbf{r} - \mathbf{r}'), \quad (5)$$

with $\varepsilon(\mathbf{r}')$ as the permittivity at position \mathbf{r}' , and q_c as the ionic charge assuming one component for simplicity. It is then assumed that the ions are neutralized by a uniform background of counter ions.

Equation (3) can now be rewritten as

$$\begin{aligned} \nabla^2 \Phi - 4\pi \beta q_c^2 \frac{1}{\varepsilon(\mathbf{r})} \rho(\mathbf{r}) \Phi &= -4\pi \frac{1}{\varepsilon(\mathbf{r}_0)} \delta(\mathbf{r} - \mathbf{r}_0), \\ h(\mathbf{r}, \mathbf{r}_0) &= -\beta q_c^2 \Phi, \end{aligned} \quad (6)$$

where Φ is the electrostatic potential. Here we have replaced \mathbf{r}_2 and \mathbf{r}_1 by \mathbf{r} and \mathbf{r}_0 , respectively. With parallel plates the number density is

$$\rho(\mathbf{r}) = \begin{cases} \rho, & z < 0, \\ 0, & 0 < z < a, \\ \rho, & a < z, \end{cases} \quad (7)$$

with equal densities (ρ constant) in the two media. By Fourier transform in the x and y directions, Eq. (6) becomes

$$\left(\frac{\partial^2}{\partial z^2} - k_\perp^2 - \kappa_\varepsilon^2 \right) \hat{\Phi} = -4\pi \frac{1}{\varepsilon(z_0)} \delta(z - z_0), \quad (8)$$

where the hat denotes Fourier transform. Furthermore, we have introduced the quantity κ_ε , defined by

$$\kappa_\varepsilon^2 = 4\pi\beta q_c^2 \rho / \varepsilon \quad (9)$$

in the two media and $\kappa_\varepsilon=0$ in the vacuum gap $0 < z < a$. Physically, κ_ε is the inverse Debye-Hückel shielding length. The symbol \mathbf{k}_\perp is the wave vector transverse to the z direction. The solution of Eq. (8) can be written in the form

$$\hat{\Phi} = 2\pi e^{q\kappa z_0} \begin{cases} \frac{1}{\varepsilon q_\kappa} e^{-q\kappa z} + B e^{q\kappa z}, & z_0 < z < 0, \\ C e^{-qz} + C_1 e^{qz}, & 0 < z < a, \\ D e^{-q\kappa z}, & a < z, \end{cases} \quad (10)$$

where $q = k_\perp$, $q_\kappa = \sqrt{k_\perp^2 + \kappa_\varepsilon^2}$. [For $z < z_0$, the $-q_\kappa(z - z_0)$ is to be replaced by $-q_\kappa|z - z_0|$ in the first line of Eq. (10).] As the boundary conditions require that $\hat{\Phi}$ and $\varepsilon \partial \hat{\Phi} / \partial z$ are continuous, one finds for the coefficient of main interest

$$D = \frac{4q e^{(q_\kappa - q)a}}{(\varepsilon q_\kappa + q)^2 (1 - A e^{-2qa})}, \quad A = \left(\frac{\varepsilon q_\kappa - q}{\varepsilon q_\kappa + q} \right)^2. \quad (11)$$

With this the pair-correlation function for free charges, for $z_0 < 0$ and $z > a$, is

$$\hat{h}(k_\perp, z, z_0) = -2\pi\beta q_c^2 D e^{-q_\kappa(z - z_0)}. \quad (12)$$

III. CASIMIR FORCE

To obtain the Casimir force the ionic interaction $\psi = \psi(r)$ in the vacuum is needed. This is the Coulomb potential $\psi = q_c^2 / r$. Its full Fourier transform is $\tilde{\psi} = 4\pi q_c^2 / k^2$, consistent with Eq. (5) with $\varepsilon = 1$. With $k^2 = k_\perp^2 + k_z^2$ this can be transformed backward to obtain (recall that $q = k_\perp$)

$$\hat{\psi}(k_\perp, z - z_0) = 2\pi q_c^2 \frac{e^{-q|z - z_0|}}{q}. \quad (13)$$

With $q^2 = k_\perp^2 = k_x^2 + k_y^2$, $dk_x dk_y = 2\pi q dq$, we now obtain for the Casimir force per unit area, for the free ions,

$$f = \frac{\rho^2}{2\pi} \int \hat{h}(q, z, z_0) \psi'_z(q, z - z_0) q dq dz_0, \quad (14)$$

where $\psi'_z = \partial \psi / \partial z$, the hat denoting Fourier transform with respect to the x and y coordinates. [We have used $\int f g dx dy = \int \hat{f} \hat{g} dk_x dk_y / (2\pi)^2$ and the translational symmetry along the xy plane.] [cf. Eq. (6) in Ref. [17]] Equation (14) follows from the general structure of the expressions for the Casimir free energy and force established in Refs. [7,9]. In those references the interaction was the dipole-dipole interaction while the correlation function was the dipole-dipole correlation function. In the present case, by contrast, those quantities are replaced by the charge-charge interaction and the charge-charge correlation function.

With $z - z_0 = u_1 + u_2 + a$, $\kappa^2 = \varepsilon \kappa_\varepsilon^2$ we obtain

$$\begin{aligned} f &= \frac{\rho^2}{2\pi} \int_0^\infty (-2\pi\beta q_c^2) D (2\pi q_c^2) \int_0^\infty \int_0^\infty \\ &\quad \times e^{-(q_\kappa + q)(u_1 + u_2 + a)} du_1 du_2 q dq \\ &= -\frac{\kappa^4}{8\pi\beta} \int_0^\infty \frac{D e^{-(q_\kappa + q)a}}{(q_\kappa + q)^2} q dq. \end{aligned} \quad (15)$$

However, this expression is missing the contribution to the force due to fluctuating dipole moments. In the evaluation of the pair-correlation function we avoided this problem by using the permittivity. Here, we may use the free charge picture of dipole moments where the induced charge density in terms of polarization is given by expression (1). Alternatively, we can use the method developed by Høye and Stell in their analysis of the ion-dipole fluid mixture where all the correlations functions between ions and dipoles were obtained for a uniform fluid [21].

The polarization of a particle at position \mathbf{r} can be written as $\mathbf{P} = \mathbf{s}_i \delta(\mathbf{r} - \mathbf{r}_i)$, where \mathbf{s}_i is the dipolar moment. The induced charge ρ_I as given by Eq. (1) will replace the q_c factors in both interaction (13) and correlation function (12). But this charge $\rho_I = -\nabla \cdot \mathbf{P}$ is distributed in space and should be convoluted with the point particle interaction $\hat{\psi}$ and the correlation function \hat{h} . Taking into account the Fourier transform in the transverse directions and the exponentials in the z direction, together with the derivative of the delta function, we find that at each end of $\hat{\psi}$ and \hat{h} the q_c should be replaced by

$$q_c \rightarrow \mathbf{h} \cdot \mathbf{s}_i, \quad \mathbf{h} = \{ik_x, ik_y, \pm q\},$$

and

$$q_c \rightarrow \mathbf{h}_\kappa \cdot \mathbf{s}_i, \quad \mathbf{h}_\kappa = \{ik_x, ik_y, \pm q_\kappa\}, \quad (16)$$

respectively. {For convenience we keep the notation of Ref. [9]. Thus this \mathbf{h} should not be confused with the correlation function h introduced in Eq. (3).} Now one needs thermal averages. There is no correlation between ionic charges and dipole moments \mathbf{s}_i for the reference system (i.e., with the ionic interaction absent). For polar or polarizable particles we assume isotropy so that

$$\langle \hat{s}_{ix}^2 \rangle = \langle \hat{s}_{iy}^2 \rangle = \langle \hat{s}_{iz}^2 \rangle = \frac{1}{3}, \quad \langle \hat{s}_{ix} \hat{s}_{iy} \rangle = 0, \dots,$$

where the hat denotes unit vectors. To obtain the Casimir force we need the average

$$Q = \langle (\mathbf{h} \cdot \hat{\mathbf{s}}_i) (\mathbf{h}_\kappa \cdot \hat{\mathbf{s}}_i)^* \rangle = \frac{1}{3} (\mathbf{h} \cdot \mathbf{h}_\kappa^*) = \frac{1}{3} (k_\perp^2 + qq_\kappa) = \frac{1}{3} q (q_\kappa + q) \quad (17)$$

($k_x^2 + k_y^2 = k_\perp^2 = q^2$). Note that this is the same as the square root of result (6.27) of Ref. [9], with q_ε replaced by q_κ . Furthermore, the dipolar density ρ_d replaces the ionic density ρ . We may now define the quantity y via

$$3y = \frac{4\pi}{3}\beta\rho_d\langle s_i^2 \rangle = 4\pi\rho_d\alpha, \quad (18)$$

where α is the polarizability. Thus for the dipoles we altogether have the replacement

$$4\pi\beta\rho q_c^2 \rightarrow 4\pi\beta\rho_d\langle s_i^2 \rangle Q = 9yQ. \quad (19)$$

This should be modified, however, by another factor $(\varepsilon - 1)/3y$, which is present in Eq. (5.5) in Ref. [9] or in Eq. (33) in Ref. [21]. This may be regarded as a contribution from correlations between neighboring dipole moments in a reference system where the ideal dipole-dipole interaction (times $-\beta$) has been subtracted from the direct correlation function. Thus we have

$$4\pi\beta\rho q_c^2 \rightarrow 3(\varepsilon - 1)Q = (\varepsilon - 1)q(q_\kappa + q). \quad (20)$$

From the free ions we have $4\pi\beta\rho q_c^2 = \kappa^2 = \varepsilon\kappa_\varepsilon^2$. Adding contribution (20) we obtain

$$\begin{aligned} \kappa^2 &\rightarrow \varepsilon\kappa_\varepsilon^2 + (\varepsilon - 1)q(q_\kappa + q) \\ &= \varepsilon(q_\kappa^2 - q^2) + (\varepsilon - 1)q(q_\kappa + q) \\ &= (\varepsilon q_\kappa - q)(q_\kappa + q). \end{aligned} \quad (21)$$

Altogether, the resulting Casimir force will be the one in which the κ^4 in Eq. (15) is replaced with the square of expression (21). We find

$$f = -\frac{1}{2\pi\beta} \int_0^\infty \frac{Ae^{-2qa}}{1 - Ae^{-2qa}} q^2 dq, \quad (22)$$

with A given by Eq. (11). This expression is a simple generalization of the situation with absence of dielectric properties [17] or absence of charges [9].

IV. FORCE BETWEEN A POLARIZABLE PARTICLE AND A HALF SPACE WITH CHARGES

This is a situation earlier considered by Pitaevskii [18]. We will reconsider it with the formalism developed above. To simplify, the polarizable particle will be considered to be confined in a thin plane or layer together with other particles of the same kind at low density. Then the electrostatic problem will be as before except for the boundary condition where now one of the half spaces (slabs) is removed and replaced with the thin layer of polarizable particles of vanishing density not influencing the electric field. Thus, instead of Eq. (10), the solution of the electrostatic problem becomes

$$\hat{\Phi} = 2\pi \begin{cases} \frac{1}{q} e^{-qz} + Be^{qz}, & 0 < z < a, \\ De^{-q\kappa z}, & a < z, \end{cases} \quad (23)$$

with the polarizable particle(s) located at $z_0=0$. From the usual boundary conditions one finds

$$D = \frac{2e^{(q_\kappa - q)a}}{\varepsilon q_\kappa + q}. \quad (24)$$

The Casimir force is obtained by a modification of Eq. (15). First, one of the integrations is reduced to a thin layer of

width Δd at $u_1=0$. Second, modification (21) is needed with dielectric media. Thus

$$4\pi\beta\rho q_c^2 = \kappa^2 \rightarrow (\varepsilon q_\kappa - q)(q_\kappa + q). \quad (25)$$

So for the half space

$$\kappa^2 \rightarrow \kappa_2^2 = (\varepsilon q_\kappa - q)(q_\kappa + q), \quad (26)$$

while for the thin layer with no free charges, $q_\kappa \rightarrow q$. Further, the permittivity ε_1 of the thin layer is directly related to the polarizability α and the density of dipolar particles $\rho_1(\rightarrow 0)$ via

$$\varepsilon_1 - 1 = 4\pi\alpha\rho_1. \quad (27)$$

So for the thin layer

$$\kappa^2 \rightarrow \kappa_1^2 = 2(\varepsilon_1 - 1)q^2 = 8\pi\alpha\rho_1 q^2. \quad (28)$$

Using Eqs. (24)–(28) in Eq. (15) we find (with $\Delta d = \int du_1$)

$$\begin{aligned} f &= -\frac{\kappa_1^2 \kappa_2^2 \Delta d}{8\pi\beta} \int_0^\infty \frac{De^{-(q_\kappa + q)a}}{q_\kappa + q} q dq \\ &= -(\rho_1 \Delta d) \frac{2\alpha}{\beta} \int_0^\infty \frac{\varepsilon q_\kappa - q}{\varepsilon q_\kappa + q} e^{-2qa} q^3 dq. \end{aligned} \quad (29)$$

Now the number of particles per unit area is $\rho_1 \Delta d$, so the force upon each of them is $f/(\rho_1 \Delta d)$. The corresponding interaction potential V is determined from $\partial V/\partial a = -f/(\rho_1 \Delta d)$. Integration of Eq. (29) thus gives

$$V = -\frac{\alpha}{\beta} \int_0^\infty \frac{\varepsilon q_\kappa - q}{\varepsilon q_\kappa + q} e^{-2qa} q^2 dq, \quad (30)$$

which is the result obtained by Pitaevskii [18]. See also the comment of Geyer *et al.*, and the reply of Pitaevskii [22].

V. FREE ENERGY OF CONCENTRIC SPHERICAL BODIES

Consider an inner sphere (ball) with radius a , and an outer sphere with inner radius $b(>a)$ and outer radius at infinity. The spheres have permittivity ε . Between the radii a and b there is a vacuum gap. So far this is the situation considered by Høye *et al.* [11]. In that paper general expressions were found, at arbitrary temperature, for nonmagnetic spheres including both the transverse magnetic (TM) and the transverse electric modes for nonzero Matsubara frequencies. In the present work we want to extend this to the situation where also free charges are present. As in the parallel-plate situation we have to restrict the evaluation to the electrostatic or the classical high-temperature case. With this limitation the resulting free energy will be a straightforward extension of previous results.

Employing spherical coordinates the solution for the potential Φ [Eq. (6)] can be written as

$$\Phi = \Phi_l(r) Y_{lm}(\theta, \varphi), \quad (31)$$

where Y_{lm} are the spherical harmonics. The radially dependent term can be written as

$$\Phi_l(r) = \begin{cases} e_\varepsilon + Bs_\varepsilon, & r < a, \\ Ce + C_1s, & a < r < b, \\ De_\varepsilon, & b < r. \end{cases} \quad (32)$$

With no free charges (i.e., $\kappa=0$), Eq. (6) is the Laplace equation, and the functions e_ε and s_ε simplify to

$$e_\varepsilon \propto e \propto \frac{1}{r^{l+1}} \quad \text{and} \quad s_\varepsilon \propto s \propto r^l. \quad (33)$$

However, with free charges present in the media the κ_ε given above in Eq. (9) will enter Eq. (6). Then the solution for the radial part becomes Riccati-Bessel functions with imaginary argument. As $\Phi_l(r)$ should be finite at the origin and zero at infinity, we can write

$$s_\varepsilon = rj_l(kr) \quad \text{and} \quad e_\varepsilon = rh_l^{(1)}(kr), \quad (34)$$

with

$$-k^2 = \kappa_\varepsilon^2 = 4\pi\beta q_c^2 \rho / \varepsilon \quad (35)$$

in the media. In vacuum $k = i\kappa_\varepsilon \rightarrow 0$, and we can put

$$e = \frac{1}{r^{l+1}} \quad \text{and} \quad s = r^l, \quad (36)$$

since proportionality factors are not needed here.

One notes that Eqs. (32) and (34) are precisely those of Ref. [11] for the situation with TM waves where Matsubara frequencies $K = -i\hbar\omega = 2\pi n / \beta$ (n integer) were used. In that case the imaginary values were $k = \omega/c$ in vacuum and $k = \sqrt{\varepsilon}\omega/c$ in the media. With this equivalence the free energy associated with the mutual interaction between the inner and outer spheres can be evaluated precisely as in Ref. [11]. Since expression (32) for Φ_l does not determine its magnitude, we can use the method in Secs. IV and V of [11] to obtain it indirectly, and thus obtain the eigenvalues of interest.

For parallel plates we were able to do this directly, through Eqs. (16)–(21) above, to obtain force (22). By integration with respect to the separation a the corresponding free energy in that case is found to be

$$\beta F = \frac{1}{2} \frac{1}{(2\pi)^2} \int \ln(1 - \lambda_q) 2\pi q dq, \quad (37)$$

which identify the eigenvalues $\lambda_q = Ae^{-2qa}$.

To obtain the corresponding eigenvalues $\lambda_{\varepsilon l}$ with the concentric spheres, we write, similarly as in [11],

$$D = \frac{D_0}{1 - \lambda_{\varepsilon l}}. \quad (38)$$

The quantity D is to be determined from the continuity of Φ_l and $\varepsilon\Phi_l'$ at the surfaces. This gives the equations

$$\begin{aligned} e_{a\varepsilon} + Bs_{a\varepsilon} &= Ce_a + C_1s_a, \\ \varepsilon(e'_{a\varepsilon} + Bs'_{a\varepsilon}) &= Ce'_a + C_1s'_a, \end{aligned}$$

$$Ce_b + C_1s_b = De_{b\varepsilon},$$

$$Ce'_b + C_1s'_b = \varepsilon De'_{b\varepsilon}, \quad (39)$$

where subscripts a and b denote the radial positions, and the prime denotes differentiation with respect to r . {Due to a calculational error in [11] the corresponding Eqs. (33) in that reference differ from those above in that ε is replaced by its inverse. The error may be related to the somewhat involved discussion in [11] on the conditions for the transverse magnetic field and the related electric field for nonzero frequencies.} Solving for D one also needs D_0 to determine $\lambda_{\varepsilon l}$. As explained in Sec. IV of [11], this can be obtained by considering single potential bonds between the two spheres. We then first remove the inner sphere by putting $a=0$ to obtain from the two last members of Eq. (39)

$$D = D_0 = c_2 C, \quad \text{with} \quad c_2 = \frac{e'_b s_b - e_b s'_b}{\varepsilon e'_{b\varepsilon} s_b - e_{b\varepsilon} s'_b}. \quad (40)$$

Then, removal of the outer sphere by putting $b=\infty$ and $C_1=0$ yields from the two first members of Eq. (39)

$$C = C_\infty = c_1, \quad \text{with} \quad c_1 = \frac{\varepsilon(e_{a\varepsilon} s'_{a\varepsilon} - e'_{a\varepsilon} s_{a\varepsilon})}{\varepsilon e_{a\varepsilon} s'_{a\varepsilon} - e'_{a\varepsilon} s_{a\varepsilon}}, \quad (41)$$

from which $D_0 = c_1 c_2$. Finally solving the full set of Eqs. (39) one finds D . This further used in Eq. (38) yields

$$\lambda_{\varepsilon l} = \frac{(\varepsilon s_a s'_{a\varepsilon} - s'_{a\varepsilon} s_{a\varepsilon})(\varepsilon e_b e'_{b\varepsilon} - e'_{b\varepsilon} e_{b\varepsilon})}{(\varepsilon e_a s'_{a\varepsilon} - e'_{a\varepsilon} s_{a\varepsilon})(\varepsilon e'_b s_b - e_{b\varepsilon} s'_b)}. \quad (42)$$

With no free charges, $\kappa_\varepsilon=0$, by which $e_\varepsilon = e = 1/r^{l+1}$ and $s_\varepsilon = s = r^l$. Then $\lambda_{\varepsilon l}$ simplifies to

$$\lambda_{\varepsilon l} = \frac{(\varepsilon - 1)^2 (l+1)l}{(\varepsilon l + l + 1)[\varepsilon(l+1) + l]} \left(\frac{a}{b}\right)^{2l+1}, \quad (43)$$

as found in Ref. [11]. The resulting free energy F is now the zero frequency ($K=0$) TM mode term of Eq. (40) in [11],

$$\beta F = \frac{1}{2} \sum_{l=1}^{\infty} (2l+1) \ln(1 - \lambda_{\varepsilon l}), \quad (44)$$

with expression (42) inserted, where k is given by Eq. (35).

VI. SUMMARY

With the use of a statistical mechanical method, the Casimir force between dielectric slabs containing free ions (i.e., semiconductors) has been evaluated in the high-temperature classical limit. Furthermore, for high temperatures the free energy of the interaction between a polarizable particle and a semiconductor slab has been obtained. Agreement with earlier results of Pitaevskii is found [18]. Finally, in the same limit the free energy of interaction between two concentric semiconducting dielectric spheres separated by a vacuum gap is found.

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